Book Review

The Universe of Quadrics

Boris Odehnal, Hellmuth Stachel, Georg Glaeser: *The Universe of Quadrics*
Springer Verlag Berlin Heidelberg, 2020, 606 p., ISBN 978-3-662-61052-7.

The Universe of Quadrics is the long-awaited oevre that complements the *The Universe of Conics* [2], a book by the same authors which has been reviewed by Robert Bix in [1]. Here, we shall consider the "Quadrics" which, as the "Conics", is a high-quality production by Springer Verlag, with more than 600 pages and over 300 beautiful color figures. (Quite ironically, at the time of writing this review, the publisher's web-page claims that the book features *one b/w illustration.*)

An important aim of the authors' when embarking on the laborious task of writing a two-volume series on conics and quadrics, both classic and well-studied topics of geometry, is the gathering of "important geometric knowledge that seems in danger of getting lost". Attentive readers can convince themselves that this danger is real indeed. It will be hard to find easily accessible alternative sources for many a topics covered by this book. This not only pertains to results directly related to quadrics but also to numerous other results and notions that are spread throughout the book. The authors deserve all credit for connecting ancient and old knowledge with brand-new results and for providing comprehensive coverage of numerous aspects of quadrics. In the past, the reviewer has been compelled to reference books from literally 100 years ago for results that he now finds in the *Universe of Quadrics.* It will certainly become a standard reference for at least the coming decades, if not longer.

The presentation of the book's contents is easily accessible. New notions and concepts are carefully introduced and explained. Most statements come with elegant proofs and, if possible, are visualized in attractive graphics. Mathematical arguments mostly rely on computations that only require basic knowledge in linear algebra and calculus. Occasionally, they are complemented by synthetic reasoning or intuitive descriptive geometry. Hardly ever advanced concepts are required so that the book is well-suited for undergraduates (while going well beyond of what can typically be taught in undergraduate courses). Reading is further facilitated by the authors' strategy to provide auxiliary information directly at places where it is first needed or fits in naturally instead of deferring it to an appendix or collecting it in a preliminary chapter. Since everything is well motivated and embedded into the surrounding texts, this concept works remarkably well and makes reading this book a pleasant experience.

The book's topic is spread over ten chapters, titled "Quadrics in Euclidean 3-Space", "Linear Algebraic Approach to Quadrics", "Projective and Affine Quadrics", "Pencils of Quadrics", "Cubic and Quartic Space Curves as Intersections of Quadrics", "Confocal Quadrics", "Special Problems", "Quadrics and Differential Geometry", "Line Geometry, Sphere Geometry, Kinematics", "Some Generalizations of Quadrics". The first three chapters study metric, affine, and projective properties of quadrics and constitute a very natural beginning. The remaining chapters are likely the result of a selection and classification process as the whole body of knowledge on quadrics is way to extensive, even for a book of 600 pages. Occasionally the authors' own research interests shine through.

Chapters 5 and 6 cover in detail pencils of quadrics and cubic and quartic space curves as intersections of quadrics. In spite of their relevance to other mathematical fields, engineering, or computer science, they are hardly every treated in that much detail. Chapter 7 introduces confocal quadrics – a very classical topic but of vital importance for modern application in mechanism science via versions of Ivory's theorem, reflections in quadrics, either in physical sense (c. f. Section 8.1) or via elliptic billiards [4]. Chapter 8 collects some special problems (reflections in quadrics, moving conics on quadrics, quadrics through skew quadrilaterals, and quadrics as rational surfaces) while Chapter 9 investigates differential geometric aspects of quadrics (curvatures, normals, curves of constant slope, geodesics, etc.)

An important application of quadrics is higher geometry where they serve as models for other geometric entities such as lines, spheres, or rigid body displacements. Certain questions of line geometry, sphere geometry, and space kinematics can be advantageously treated in these models but require a good understanding of the more-dimensional geometry of quadrics and its relation to the geometry of three space via geometric/algebraic mappings. These aspects are discussed in detail in Chapter 10. It will not rival monographs on line geometry, sphere geometry or the Study model of space kinematics but provides useful introductions to these topics. The concluding Chapter 11 is devoted to diverse generalizations of quadrics like Müller's cubic surface and Plücker's conoid, superquadrics, surfaces of osculating circles and surfaces defined by constraints on sums or products of distances to fixed points.

An important aspect of this book that should not go by unnoticed is its pictures. All three authors are well-known for their unique visualization style and the high quality of their images (the reader is invited to confirm this observation at hand of the articles [3, 4] which appear in this issue of the Journal for Geometry and Graphics). Their pictures are not only of exceptional graphical quality with a lot of attention on small details, they also exhibit high didactic value. Illustrations are never done casually and one can sense the amount of consideration and work that went into each of them.

The "Universe of Quadrics" provides a comprehensive, unified and accessible treatment of the subject. It is therefore valuable for diverse audiences: Students on all levels of academic education can profit from lucid and easily accessible explanations and great illustrations. A suitable selection of chapters or topics could be transformed into a university lecture on quadrics at graduate level while bits and pieces can be used also in undergraduate lectures. For researchers in the field of geometry, it will provide an indispensable resource and valuable reference.

References

[1] R. Bix: *Review of "The Universe of Conics. From the ancient Greeks to 21st century developments."* J. Geom. Graphics **20**(2), 285, 2016.

[2] G. Glaeser, H. Stachel, and B. Odehnal: *The Universe of Conics. From the ancient Greeks to 21st century.* Springer Spektrum, 2016. ISBN 978-3-662-45449-7. doi: 10.1007/978-3-662-45450-3.

[3] B. Odehnal: *Isoptic Ruled Surfaces of Developable Surfaces.* J. Geom. Graph. **25**(1), 1–17, 2021.

[4] H. Stachel: *Isometric Billiards in Ellipses and Focal Billiards in Ellipsoids.* J. Geom. Graph. **25**(1), 97–118, 2021.

Hans-Peter Schröcker
Department of Basic Sciences in Engineering Sciences
University of Innsbruck, Austria
Email: hans-peter.schroecker@uibk.ac.at